Stochastic continuity

(1) Continuous in probability

A random process $\{X(t)\}$ is called continuous in probability at t if for any $\varepsilon > 0$,

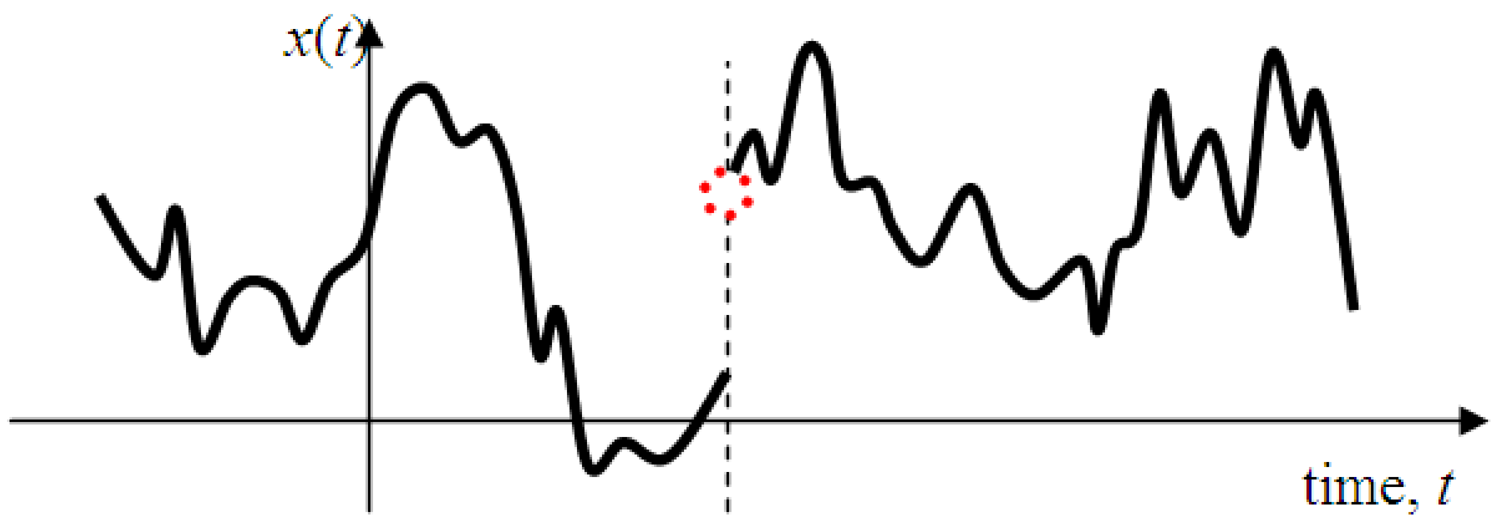
$$P\left(|X(t+h) - X(t)| > \varepsilon\right) \underset{h \to 0}{\longrightarrow} 0$$

(2) Continuous in mean-square sense

A random process $\{X(t)\}$ is called mean-square (MS) continuous at t if

$$E\left\{\left[X(t+\varepsilon) - X(t)\right]^2\right\} \underset{\varepsilon \to 0}{\longrightarrow} 0$$

Is it possible to have a *realization* which is discontinuous at time t whereas the random process is mean-square continuous?



A random process $\{X(t)\}$ is MS continuous if its autocorrelation function is continuous.

Proof

$$E\{[X(t+\varepsilon)-X(t)]^2\}=R(t+\varepsilon,t+\varepsilon)-2R(t+\varepsilon,t)+R(t,t)$$

If $R(t_1,t_2)$ is continuous, then the RHS approaches zero as $\varepsilon\to 0$. [Note that $R(t_1,t_2)$ is used in the above equation to simply the expression.]

Suppose that MS continuous property holds for every t in an interval I. It follows that almost all samples (or realizations) of $\{X(t)\}$ will be continuous for a particular point of I. It does not follow, however, that these samples (or realizations) of $\{X(t)\}$ will be continuous for every point in I.

If $\{X(t)\}$ is MS continuous, then its mean is continuous, i.e.,

$$m_X(t+\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} m_X(t)$$

Proof

$$E\{[X(t+\varepsilon) - X(t)]^2\} \geq \{E[X(t+\varepsilon) - X(t)]\}^2$$

Therefore, $E[X(t+\varepsilon) - X(t)] \underset{\varepsilon \to 0}{\longrightarrow} 0$

[Note: $Var(X) = E(X^2) - [E(X)]^2 \geq 0$]

(3) Almost-surely continuous

A random process $\{X(t)\}$ is called almost-surely continuous at t if

$$P\left(\omega : \lim_{h \to 0} | X(t+h;\omega) - X(t;\omega) | = 0\right) = 1$$

If $\{X(t)\}$ is continuous in probability (mean-square continuous, almost-surely continuous) at every t, then it is said to be continuous in probability (mean-square continuous, almost-surely continuous).

Stochastic Convergence

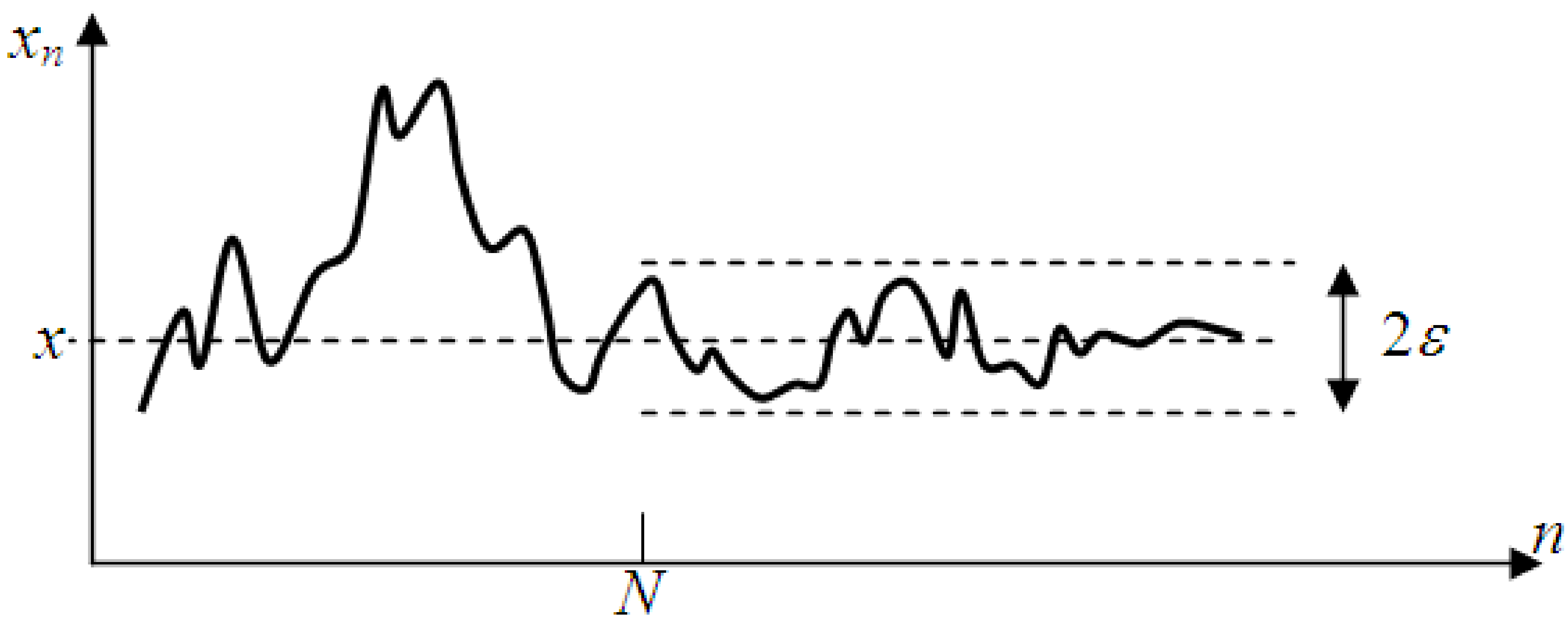
- **A *random sequence* or a discrete-time random process is a sequence of random variables $\{X_1(\omega), X_2(\omega), \ldots, X_n(\omega), \ldots\} = \{X_n(\omega)\}$, $\omega \in \Omega$.**
- **For a specific ω, $\{X_n(\omega)\}$ is a sequence of numbers that might or might not converge. The notion of convergence of a random sequence can be given several interpretations.**

Sure convergence (convergence everywhere)

- **The sequence of random variables $\{X_n(\omega)\}$ converges surely to the random variable $X(\omega)$ if the sequence of functions $X_n(\omega)$ converges to $X(\omega)$ as $n \to \infty$ for all $\omega \in \Omega$, i.e.,**

$$X_n(\omega) \to X(\omega) \quad \text{as } n \to \infty \quad \text{for all } \omega \in \Omega.$$

A sequence of real numbers x_n converges to the real number x if, given any $\varepsilon > 0$, we can always specify an integer N such that for all values of n beyond N we can guarantee that $|x_n - x| < \varepsilon$.



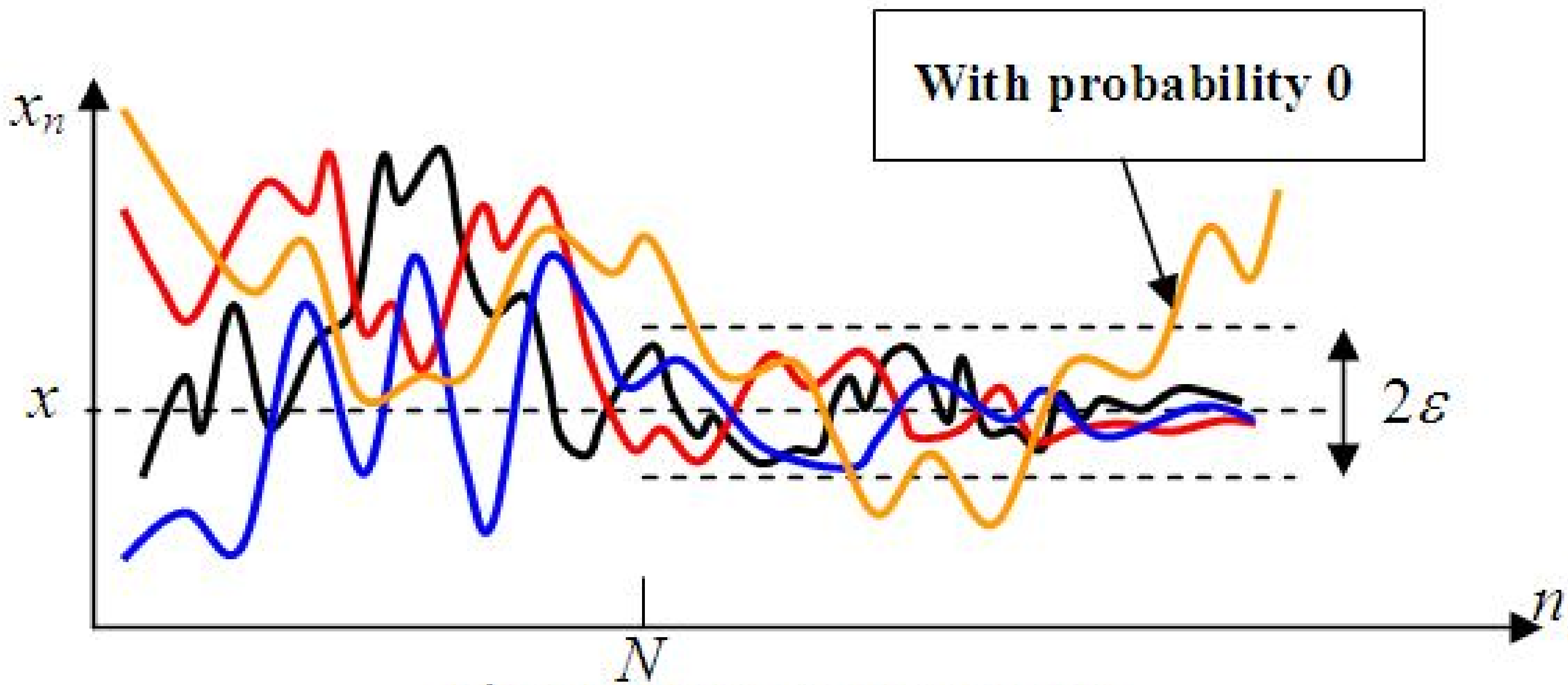
Convergence of a sequence of numbers

Sure convergence requires that the sample sequence corresponding to every ω converges. However, it does not require that all the sample sequences converge to the same value; that is the sample sequences for different ω *and* ω' can converge to different values.

Almost-sure convergence (Convergence with probability 1)

The sequence of random variables $\{X_n(\omega)\}$ converges almost surely to the random variable $X(\omega)$ if the sequence of functions $X_n(\omega)$ converges to $X(\omega)$ as $n \to \infty$ for all $\omega \in \Omega$, except possibly on a set of probability zero; i.e.,

$$P\left[\omega : X_n(\omega) \underset{n\to\infty}{\longrightarrow} X(\omega)\right] = 1.$$



Almost-sure convergence

Mean-square convergence

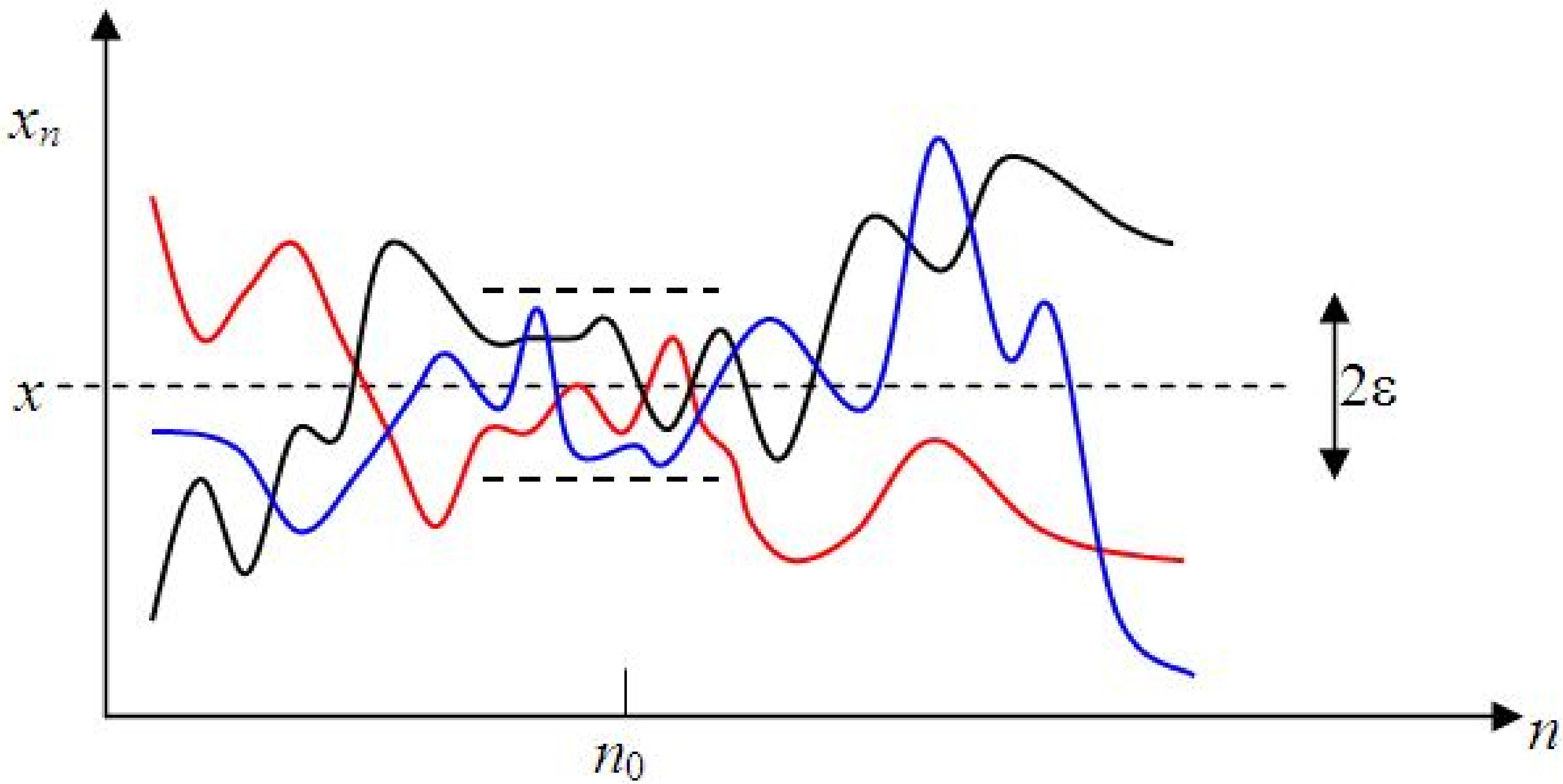
The sequence of random variables $\{X_n(\omega)\}$ converges in the mean square sense to the random variable $X(\omega)$ if

$$E\left[\left(X_n(\omega) - X(\omega)\right)^2\right] \to 0 \quad \text{as} \quad n \to \infty$$

Convergence in probability

The sequence of random variables $\{X_n(\omega)\}$ converges in probability to the random variable $X(\omega)$ if, for any $\varepsilon > 0$,

$$P\left[\left|X_n(\omega) - X(\omega)\right| > \varepsilon\right] \to 0 \quad \text{as} \quad n \to \infty$$



Convergence in probability for the case where the limiting random variable is a constant x

Convergence in distribution

The sequence of random variables $\{X_n(\omega)\}$ with cumulative distribution functions $\{F_n(x)\}$ converges in distribution to the random variable $X(\omega)$ with cumulative distribution functions $F(x)$ if

$$F_n(x) \to F(x) \quad \text{as} \quad n \to \infty$$

for all x at which $F(x)$ is continuous.

Remarks

- **Convergence with probability one applies to the individual realizations of the random process. Convergence in probability does not.**
- **The weak law of large numbers is an example of convergence in probability.**
- **The strong law of large numbers is an example of convergence with probability 1.**
- **The central limit theorem is an example of convergence in distribution.**

Weak Law of Large Numbers (WLLN)

Let f(·) be a density with finite mean μ and finite variance. Let $\overline{X}_n$ be the sample mean of a random sample of size n from f(·), then for any $\varepsilon > 0$,

$$P[-\varepsilon < \overline{X}_n - \mu < \varepsilon] \to 1 \quad \text{as} \quad n \to \infty$$

Strong Law of Large Numbers (SLLN)

Let f(·) be a density with finite mean μ and finite variance. Let $\overline{X}_n$ be the sample mean of a random sample of size n from f(·), then for any $\varepsilon > 0$,

$$P\left[\lim_{n\to\infty} \overline{X}_n = \mu\right] = 1$$

The Central Limit Theorem

Let f(·) be a density with mean μ and finite variance σ^2.

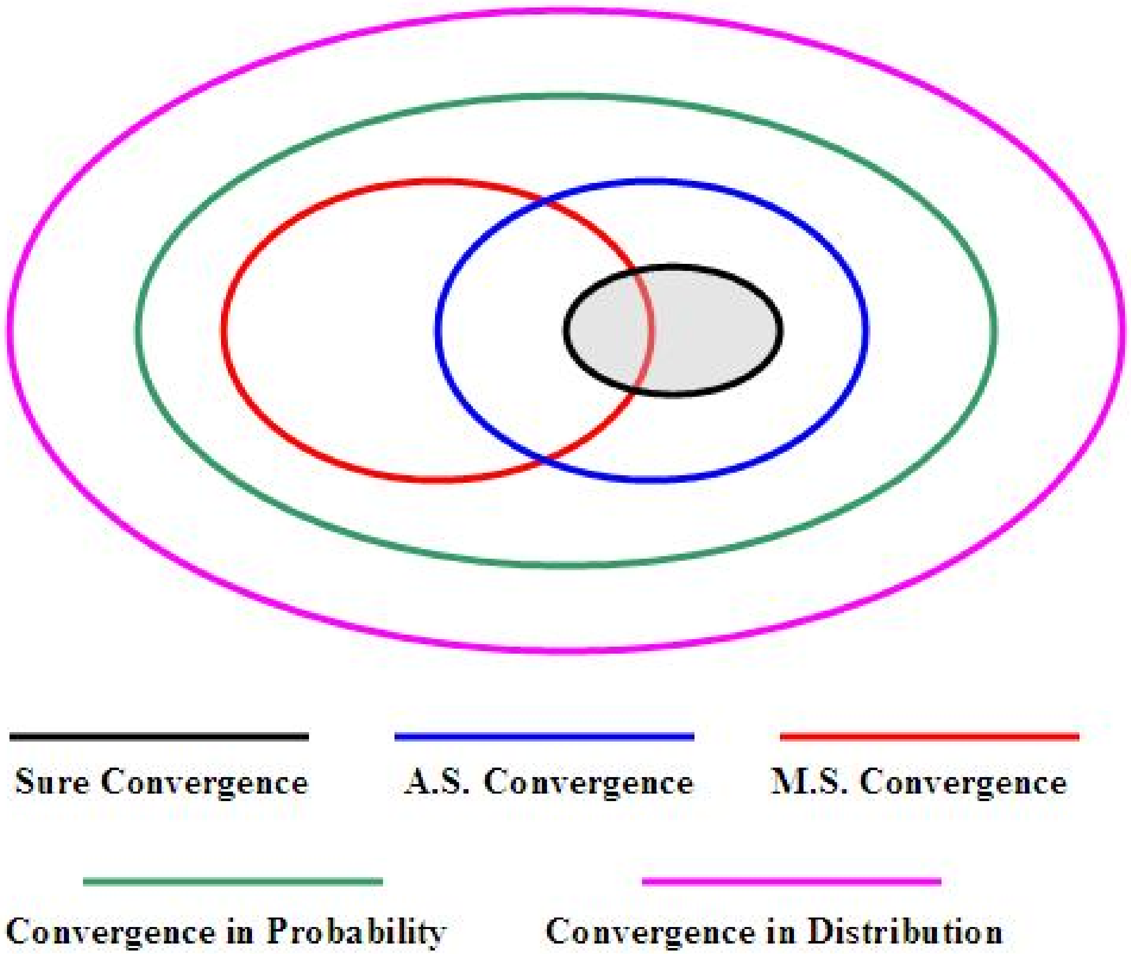
Let $\overline{X}_n$ be the sample mean of a random sample of size n

from f(·). Then

$$Z_n = \frac{\overline{X}_n - \mu}{\sigma / \sqrt{n}}$$

approaches the standard normal distribution as n approaches infinity. [Note: It is equivalent to say that $\overline{X}_n$ approaches a normal distribution with expected value μ and variance σ^2/n as n approaches infinity.]

Venn diagram of relation of types of convergence



Note that even sure convergence may not imply mean square convergence.

Example

Let ω be selected at random from the interval $S = [0, 1]$, where we assume that the probability that ω is in a subinterval of S is equal to the length of the subinterval. For $n = 1, 2, \ldots$ we define the following five sequences of random variables:

$$U_n(\omega) = \omega / n, \quad V_n(\omega) = \omega\left(1 - \frac{1}{n}\right), \quad W_n(\omega) = \omega \cdot e^n,$$

$$Y_n(\omega) = \cos 2n\pi\omega, \quad Z_n(\omega) = e^{-n(n\omega - 1)}$$

Determine the stochastic convergence of these random sequences and identify the limiting random variable.

$U_n(\omega) \underset{n\to\infty}{\to} U(\omega) = 0$ for every $\omega \in S$. Therefore, it converges surely to a constant 0.

$V_n(\omega) \underset{n\to\infty}{\to} V(\omega) = \omega$ for every $\omega \in S$. Therefore, it converges surely to a random variable which is uniformly distributed over [0, 1].

$$E\left[(V_n(\omega) - \omega)^2\right] = E\left[\left(\frac{\omega}{n}\right)^2\right] = \int_0^1 \frac{\omega^2}{n^2} d\omega = \frac{1}{3n^2}.$$

$E\left[(V_n(\omega) - \omega)^2\right] \underset{n\to\infty}{\to} 0$. Thus, the sequence $V_n(\omega)$ converges in the mean-square sense.

$W_n(\omega)$ converges to 0 for $\omega = 0$, but diverges to infinite for all other values of ω. Therefore, it does not converge.

$Y_n(\omega)$ converges to 1 for $\omega = 0$ and $\omega = 1$, but oscillates between -1 and 1 for all other values of ω. Therefore, it does not converge.

$$Z_n(\omega = 0) = e^n \underset{n\to\infty}{\longrightarrow} +\infty, \quad Z_n(\omega) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for } \omega > (1/n).$$

$P[\omega > 0] = 1$. Thus, $Z_n(\omega)$ converges almost surely to 0.

$$E\left[(Z_n(\omega) - 0)^2\right] = E\left[e^{-2n(n\omega - 1)}\right] = e^{2n}\int_0^1 e^{-2n^2\omega}\,d\omega = \frac{e^{2n}}{2n^2}\left(1 - e^{-2n^2}\right).$$

As n approaches infinity, the rightmost term in the above equation approaches infinity. Therefore, the sequence $Z_n(\omega)$ does not converge in the mean square sense even though it converges almost surely.

Ergodic Theorem

A discrete time random process $\{X_n, n = 0, 1, 2, \ldots\}$ is said to satisfy an ergodic theorem if there exists a random variable X such that in some sense

$$\sum_{i=0}^{n-1} X_i / n \underset{n\to\infty}{\longrightarrow} X$$

The type of convergence determines the type of the ergodic theorem. For example, if the convergence is in mean square sense, the result is called a mean ergodic theorem. If the convergence is with probability one, it is called an almost sure ergodic theorem.

A continuous time random process $\{X(t)\}$ is said to satisfy an ergodic theorem if there exists a random variable X such that

$$\frac{1}{T}\int_0^T X(t)dt \underset{T\to\infty}{\longrightarrow} X$$

where again the type of convergence determines the type of the ergodic theorem.

Note that we only require the time average to converge, however, it does not need to converge to some constant, for example the common expectation of the random process. In fact, ergodic theorem can hold even for nonstationary random processes where $E[X(t)]$ does depend on time t.

The Mean-Square Ergodic Theorem

Let $\{X_n\}$ be a random process with mean function $E[X_n]$ and covariance function $C_X(k,j)$. (The process need not to be even weakly stationary.) Necessary and sufficient conditions for the existence of a constant m such that

$$E\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i - m\right)^2\right] \to 0 \quad \text{as} \quad n \to \infty$$

are that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} E(X_i) = m\,, \quad \lim_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n} C_X(i,k) = 0\,.$$

The above theorem shows that one can expect a sample average to converge to a constant in mean square sense if and only if the average of the means converges and if the memory dies out asymptotically, that is , if the covariance decreases as the lag increases.

Mean-Ergodic Processes

A random process $\{X(t)\}$ with constant mean μ is said to be mean-ergodic if it satisfies

$$P\left\{\frac{1}{2T}\int_{-T}^{T} x(t)dt \underset{T\to\infty}{\longrightarrow} \mu\right\} = 1.$$

Strong or Individual Ergodic Theorem

Let $\{X_n\}$ be a strictly stationary random process with $E[X_n] < \infty$. Then the sample mean $\sum_{i=1}^{n} X_i / n$ converges to a limit with probability one.

Let $\{X(t)\}$ be a wide-sense stationary random process with constant mean μ and covariance function $C(t)$. Then $\{X(t)\}$ is mean-ergidic if and only if

$$\frac{1}{2T}\int_{-2T}^{2T} C(\tau)\left(1-\frac{|\tau|}{2T}\right)d\tau \underset{T\to\infty}{\longrightarrow} 0, \text{ or equivalently,}$$

$$\frac{1}{T}\int_{0}^{2T} C(\tau)\left(1-\frac{\tau}{2T}\right)d\tau \underset{T\to\infty}{\longrightarrow} 0$$

- Let $\{X(t)\}$ be a wide-sense stationary random process with constant mean μ and covariance function $C(t)$ and $\int_{-\infty}^{\infty} |C(\tau)|\, d\tau < \infty$, then $\{X(t)\}$ is mean-ergidic.
- Let $\{X(t)\}$ be a wide-sense stationary random process with constant mean μ and covariance function $C(t)$ and $C(0) < \infty$ and $C(\tau) \xrightarrow[|\tau| \to \infty]{} 0$, then $\{X(t)\}$ is mean-ergidic.

Examples of Stochastic Processes

- ***iid random process***

A discrete time random process $\{X(t), t = 1, 2, \ldots\}$ is said to be independent and identically distributed (*iid*) if any finite number, say k, of random variables $X(t_1)$, $X(t_2)$, …, $X(t_k)$ are mutually independent and have a common cumulative distribution function $F_X(\cdot)$.

- **The joint cdf for $X(t_1), X(t_2), \ldots, X(t_k)$ is given by**

$$F_{X_1, X_2, \cdots, X_k}(x_1, x_2, \cdots, x_k) = P\left(X_1 \le x_1, X_2 \le x_2, \cdots, X_k \le x_k\right)$$

$$= F_X(x_1) F_X(x_2) \cdots F_X(x_k)$$

- **It also yields**

$$p_{X_1, X_2, \cdots, X_k}(x_1, x_2, \cdots, x_k) = p_X(x_1) p_X(x_2) \cdots p_X(x_k)$$

where $p(x)$ represents the common probability mass function.

Let $\{X_n, n = 0, 1, 2, \ldots\}$ be a sequence of independent Bernoulli random variables with parameter p. It is therefore an iid Bernoulli random process and $E[X_n] = p$ and $Var[X_n] = p(1-p)$.